MID-TERM EXAM ISI BANGALORE SEPTEMBER 2015

M.MATH LINEAR ALGEBRA 100 POINTS

Notes.

(a) Begin each answer on a separate sheet and ensure that the answers to all the parts to a question are arranged contiguously.

(b) Assume only those results that have been proved in class. All other steps should be justified.

(c) $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers $\mathbb{C}$ = complex numbers.

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1. [15 points] For the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ find a matrix P such that PAP^{-1} is diagonal and give a formula for A^{30}.

2. [15 points] Let F be a field and $n > 0$ an integer. Give an example of two $n \times n$ matrices A, B that are not similar, but whose characteristic polynomials are equal.

3. [15 points] Let F be a field. Given two linearly independent collections of vectors $v_1, \dots, v_r \in F^n$ and $w_1, \dots, w_r \in F^n$, show that there exists an invertible $n \times n$ matrix A such that $Av_i = w_i$ for $i \leq r$.

4. [20 points]
Let A be an $m \times n$ matrix and let X be an invertible $m \times m$ matrix. Set $A' = XA$.
   (i) Show that the columns $C_{i_1}, \dots, C_{i_r}$ of A are linearly independent iff the corresponding columns $C'_{i_1}, \dots, C'_{i_r}$ of A' are so. Likewise, show that $C_{i_1}, \dots, C_{i_r}$ span the column space of A iff the corresponding ones of A' have the same property.
   (ii) Deduce that if the pivot entries in (reduced) row echelon form of A are in columns indexed by $j_1, \dots, j_r$, then the corresponding columns $C_{j_1}, \dots, C_{j_r}$ of A form a basis of the column space of A.

5. [15 points] Show that the equation $AB - BA = I$ has no solution in $n \times n$ matrices A, B over $\mathbb{R}$. Show that over the finite field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, there is a solution for A, B for $n = 2$.

6. [20 points] Let $v_1, \dots, v_r$ be mutually orthogonal (nonzero) vectors in $\mathbb{R}^n$.
   (i) If W is the subspace $\{w \in \mathbb{R}^n | v_i \cdot w = 0 \, \forall i\}$ and $V = \mathrm{Span}(v_1, \dots, v_r)$, then show that dimension of W is $n - r$ and that $V \oplus W = \mathbb{R}^n$.
   (ii) Show that any collection of mutually orthogonal vectors in $\mathbb{R}^n$ is a part of a basis consisting of mutually orthogonal vectors.